Difference Equations (OR Recurrence Relations)

Reference 20.1 K. Sydsaeter and P. Hammond

Difference equation is a discrete time version of the differential equations. It shows how does a variable in time period t related to its values in time period t-1, t-2 and so on. Another way to see the equation as to show the change in a particular variable x from one year to another.

We will only be discussing the first order difference equations.

$$x_t = f(t, x_{t-1}) \qquad (t = 1, 2, 3, \text{---})$$

Or $\Delta x_t = g(t, x_{t-1})$ where $\Delta x_t = x_t - x_{t-1}$

Both equations are similar as second equation can be written in the form of first equation.

$$\Delta x_t = x_t - x_{t-1} = g(t, x_{t-1})$$

$$x_t = x_{t-1} + g(t, x_{t-1}) = f(t, x_{t-1})$$

Redefine

Our aim to find a function of x_t which depends only on time t alone.

So. we need to get rid of x_{t-1}. This is also called the solution of the difference equation.

We will be using the Iterative method to solve the difference equation.

Let us start with the simplest difference equation and then we will generalize.

Let at $t=0$, $x=x_0$

(1) $x_t = x_{t-1} \quad \forall t$ or $\Delta x_t = 0$

This equation implies variable x doesnot change over time.

Solution

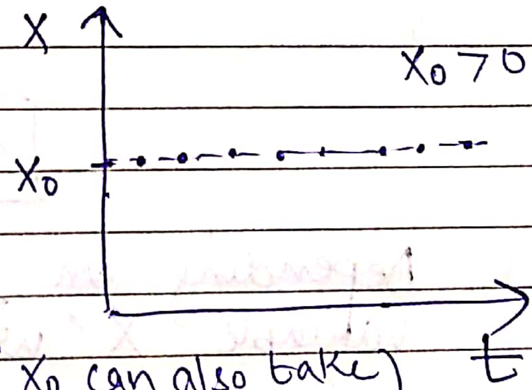
$x_1 = x_0$

$x_2 = x_1 = x_0$

$x_3 = x_2 = x_1 = x_0$

$\vdots$

$$\boxed{x_t = x_0}$$



(X_0 can also take negative value)

(2) $x_t = x_{t-1} + b$

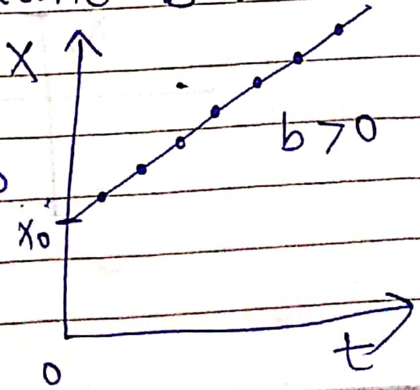
Value of x in time t is equal to preceding year quantity plus a constant b.

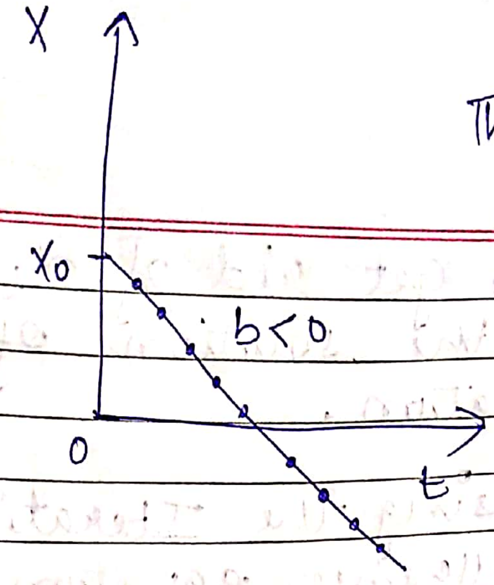
So. $x_1 = x_0 + b$

$x_2 = x_1 = (x_0 + b) + b = x_0 + 2b$

$\vdots$

$$\boxed{x_t = x_0 + bt}$$





The graph won't be a smooth curve as time is discrete. (X_0 can also take negative value)

(3) $x_t = a x_{t-1} \quad \forall t$

Each year the value / quantity will be a times the value in the preceding year.

Solution : $X_1 = a X_0$

$X_2 = aX_1 = a(aX_0) = a^2 X_0$

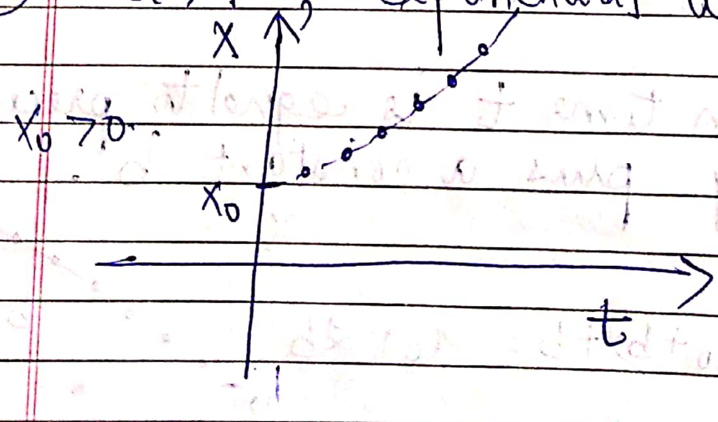
$X_3 = aX_2 = a(a^2 X_0) = a^3 X_0$

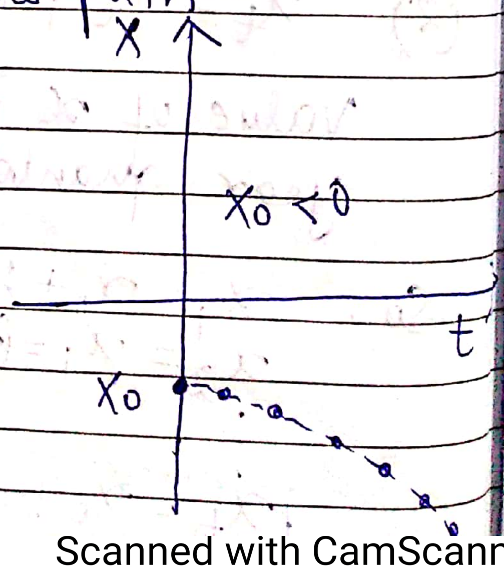
$\vdots$

$$\boxed{X_t = a^t X_0}$$

Depending on value of a, time path of variable X will vary.

(a) $a > 1$; exponential time path

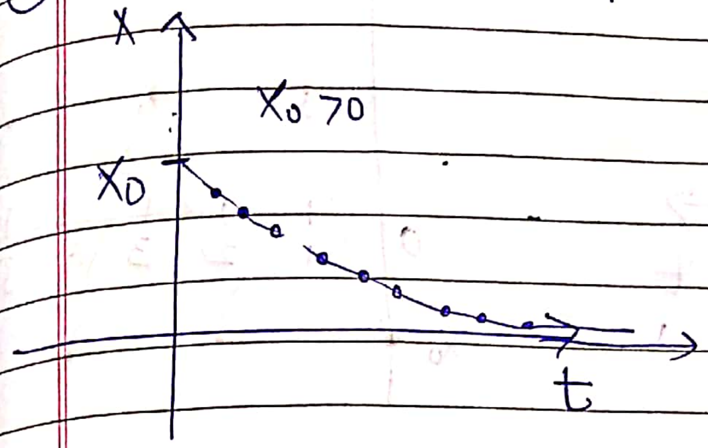


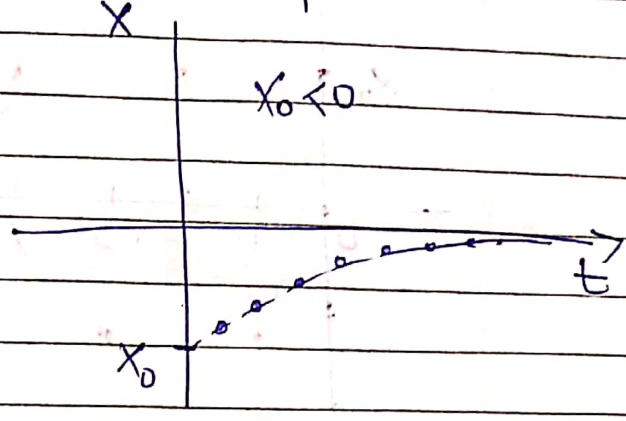


Date
Page

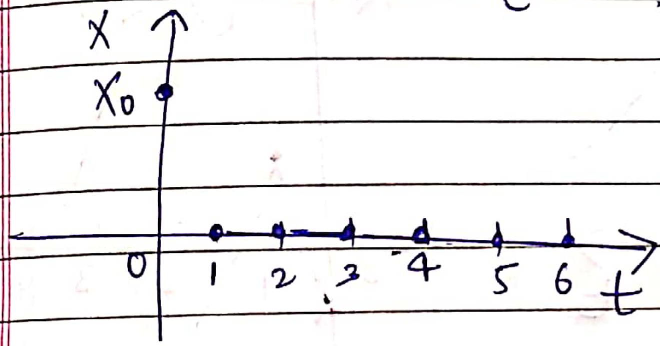
(b) $a = 1$; we have already solved this case

(c) $0 < a < 1$; exponential time path.





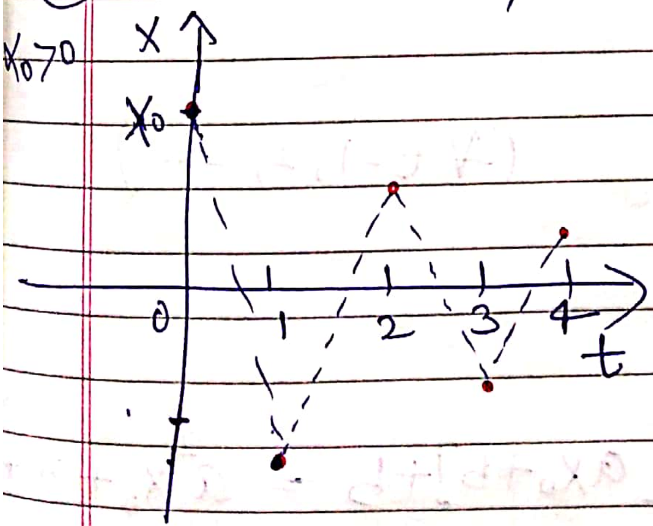
(d) $a = 0 \Rightarrow X_t = 0 \ \forall t = 1, 2, \ldots$ So, if $X_0 \neq 0$.

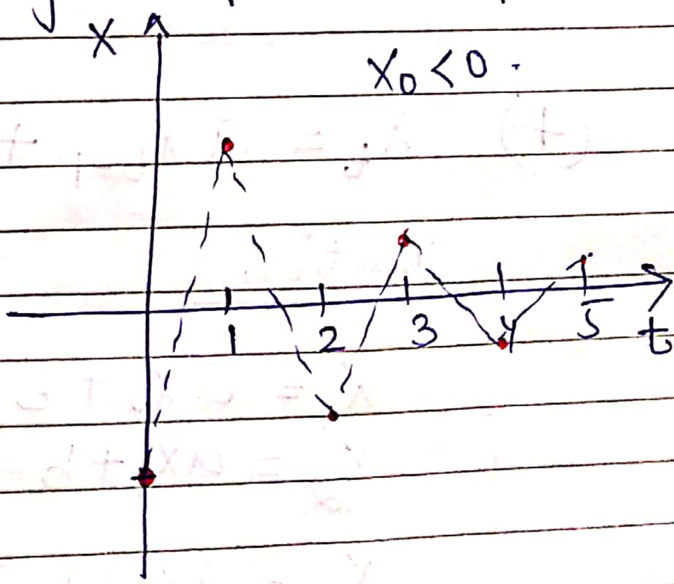


At $t = 0 \quad X = X_0$

$\forall t \neq 0 \quad X = 0$.

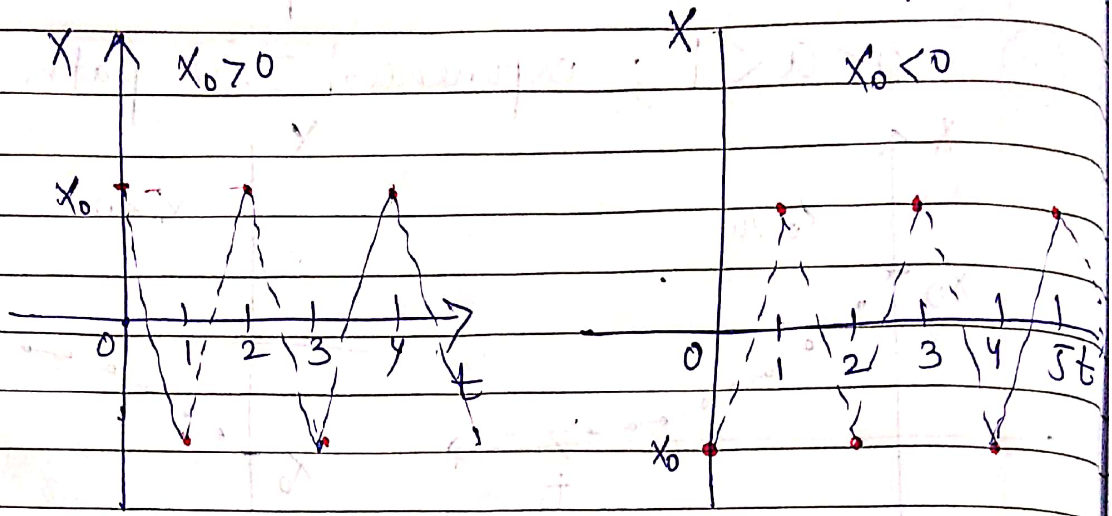
(e) $-1 < a < 0$; oscillatory time path (damped)



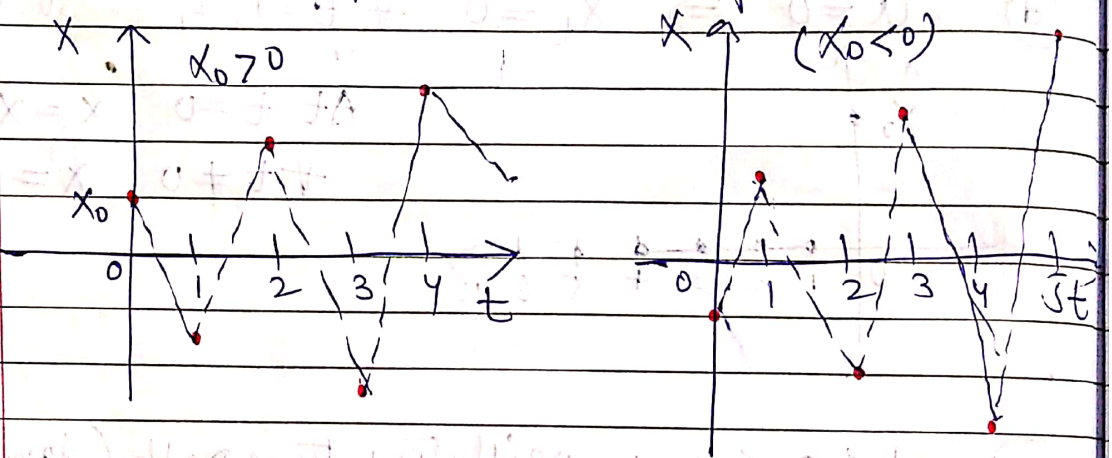


Scanned with CamScanner

(f) $a = -1 \rightarrow$ equal oscillatory time path.



(g) $a < -1 \rightarrow$ explosive oscillatory time path.



(4) $X_t = aX_{t-1} + b$. $\quad (\forall\, t = 1, 2, \text{---})$

Solution

$X_1 = aX_0 + b$.

$X_2 = aX_1 + b = a[aX_0 + b] + b = a^2 X_0 + ab + b$

$X_3 = aX_2 + b = a[a^2 X_0 + ab + b] + b$

$X_3 = a^3 X_0 + a^2 b + ab + b$

and so on.

Page

$$x_t = a^t x_0 + a^{t-1} b + a^{t-2} b + \cdots + a b + b$$

$$x_t = a^t x_0 + b\left[\underbrace{a^{t-1} + a^{t-2} + \cdots + a + 1}\right]$$

finite Geometric series with common factor a.

$$x_t = a^t x_0 + b\left[\frac{1-a^t}{1-a}\right] \qquad (a \neq 1)$$

$$x_t = a^t x_0 + \frac{b}{1-a} - \frac{b a^t}{1-a}.$$

$$\boxed{x_t = a^t\left[x_0 - \frac{b}{1-a}\right] + \frac{b}{1-a}} \quad a \neq 1$$

For $a=1$, we have already solved before.

We will see the graph of x_t a little later.

General case

(5) $x_t = a x_{t-1} + b_t \qquad (t = 1, 2, \ldots, n)$

Solution: $x_1 = a x_0 + b_1$

$x_2 = a x_1 + b_2 = a[a x_0 + b_1] + b_2$

$= a^2 x_0 + a b_1 + b_2$.

$x_3 = a x_2 + b_3 = a[a^2 x_0 + a b_1 + b_2] + b_3$.

$= a^3 x_0 + a^2 b_1 + a b_2 + b_3$

and so on.

$$\boxed{X_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_k}$$

For any difference equation, we can check its solution by putting it back in the difference equation.

Let's check for the general case.

$$X_t = a x_{t-1} + b_t$$

Acc to solution we have got

$$X_{t-1} = a^{t-1} x_0 + \sum_{k=1}^{t-1} a^{(t-1)-k} b_k .$$

Put value of X_{t-1}

$$\text{So. } X_t = a \left[a^{t-1} x_0 + \sum_{k=1}^{t-1} a^{(t-1)-k} b_k \right] + b_t$$

$$X_t = a^t x_0 + \sum_{k=1}^{t-1} a^{t-k} b_k + b_t$$

when $k = t$, $a^{t-k} b_k = b_t$

$$\text{So. } \boxed{X_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_k}$$

From this solution, we can get all the previous cases by putting different values of a and b.

classmate

Date

Page

example $a = 1$ and $b_t = 0$ $\forall t$

So. $X_t = X_{t-1}$ our difference equation.

From the general solution $a^t = 1$.

$\sum_{k=1}^{t} a^{t-k} b_k = 0$ since $b_t = 0$

So. $\boxed{x_t = x_0}$ solution.

Examples

(i) $x_t = \frac{1}{2} x_{t-1} + 3$ Let $X_0 = 5$

Here $a = \frac{1}{2}$, $b = 3$; Refer to difference equation 4

if $x_t = a x_{t-1} + b \iff x_t = a^t \left[x_0 - \frac{b}{1-a} \right] + \frac{b}{1-a}$

So. solution: $x_t = \left(\frac{1}{2}\right)^t \left[5 - \frac{3}{1-\frac{1}{2}} \right] + \frac{3}{1-\frac{1}{2}}$

$x_t = \left(\frac{1}{2}\right)^t [-1] + 6$.

Solution $\boxed{x_t = -\left(\frac{1}{2}\right)^t + 6}$

(ii) $\Delta x_t = 2x_t - 9$ $\quad x_0 = 5$

$x_t - x_{t-1} = 2x_t - 9$.

or $x_t = -x_{t-1} + 9$.

$a = -1 \quad b = 9$.

Solution $x_t = (-1)^t \left[5 - \frac{9}{1-(-1)}\right] + \frac{9}{1-(-1)}$

$x_t = (-1)^t \left[\frac{1}{2}\right] + 9/2$.

Model of Growth

$S_t = \alpha Y_t \ ; \ I_t = \beta(Y_t - Y_{t-1})$

Equilibrium condition $\rightarrow$ $S_t = I_t$. $\quad (\beta > \alpha > 0)$.

Y_t denote national income, $I_t \rightarrow$ total investment, $S_t \rightarrow$ Total savings $\rightarrow$ in time period t.

We need to deduce a difference equation determining the path of Y_t, given Y_0 and solve it.

$\alpha Y_t = \beta[Y_t - Y_{t-1}]$

$\Rightarrow Y_t = \left(\frac{\beta}{\beta - \alpha}\right) Y_{t-1}$ $\quad \left[\begin{array}{l} a = \beta/\beta-\alpha \\ b = 0. \end{array}\right.$

Solution $Y_t = \left(\frac{\beta}{\beta-\alpha}\right)^t Y_0$.

Equilibrium and stability

Consider the difference equation

$$X_t = aX_{t-1} + b$$

$(a \neq 1)$

Its solution is: $X_t = a^t\left[X_0 - \frac{b}{1-a}\right] + \frac{b}{1-a}$

Suppose at time period $t=0$, $X_0 = \frac{b}{1-a}$.

then from the solution we can see that $X_t = \frac{b}{1-a}$ $\forall t$ [remain constant]

Let $X_0 \neq \frac{b}{1-a}$, But at some time period s; $x_s = \frac{b}{1-a}$, then value in period $s+1$, $s+2$ will also be $\frac{b}{1-a}$

$$x_{s+1} = ax_s + b = a\left(\frac{b}{1-a}\right) + b = \frac{b}{1-a}$$

$$x_{s+2} = ax_{s+1} + b = a\left(\frac{b}{1-a}\right) + b = \frac{b}{1-a}.$$

and so on.

We have seen that if x_s ever becomes equal to $\frac{b}{1-a}$ at some time s, then x_t will remain at this constant level for all $t \geq s$ so,

$x^* = \frac{b}{1-a}$ is called an equilibrium (or stationary state)

So once we reach $x^* = \frac{b}{1-a}$, we will stay at x^*. So equilibrium has the property that $x_t = x^* \ \forall t$.

So from difference equation as
$x_t = x_{t-1} = x^*$

So, $x_t = ax_{t-1} + b \Rightarrow x^* = ax^* + b$
or $x^* = \frac{b}{1-a} \quad (a \neq 1)$

So. solution can be rewritten as

$$x_t = a^t [x_0 - x^*] + x^* \qquad [a \neq 1]$$

When $a = 1$ $\quad x_t = x_{t-1} + b$.

Solution: $\quad x_t = x_0 + bt$

For equilibrium $x_t = x_{t-1} \ \forall t$ but as long as $b \neq 0$, this difference equation has no equilibrium.

If $b = 0$, $x_t = x_{t-1}$ so $\boxed{x_t = x_0}$ is the equilibrium or solution

Stability of Equilibrium

If we start at $x^* = \frac{b}{1-a}$, we stay at $x^* = \frac{b}{1-a}$. But the problem is we don't start at $x^* = \frac{b}{1-a}$. So we need to know

are there forces which will move us towards x^* or move us away from x^* ~~from the~~ pt $x_0 \neq \frac{b}{1-a}$.

- $x_t = a^t [x_0 - x^*] + x^*$ → This equation gives us the time path of x_t. Since $x_0 - x^*$ and x^* ~~is~~ are constants

So what happens to x_t as $t \to \infty$ depends on the function a^t alone.

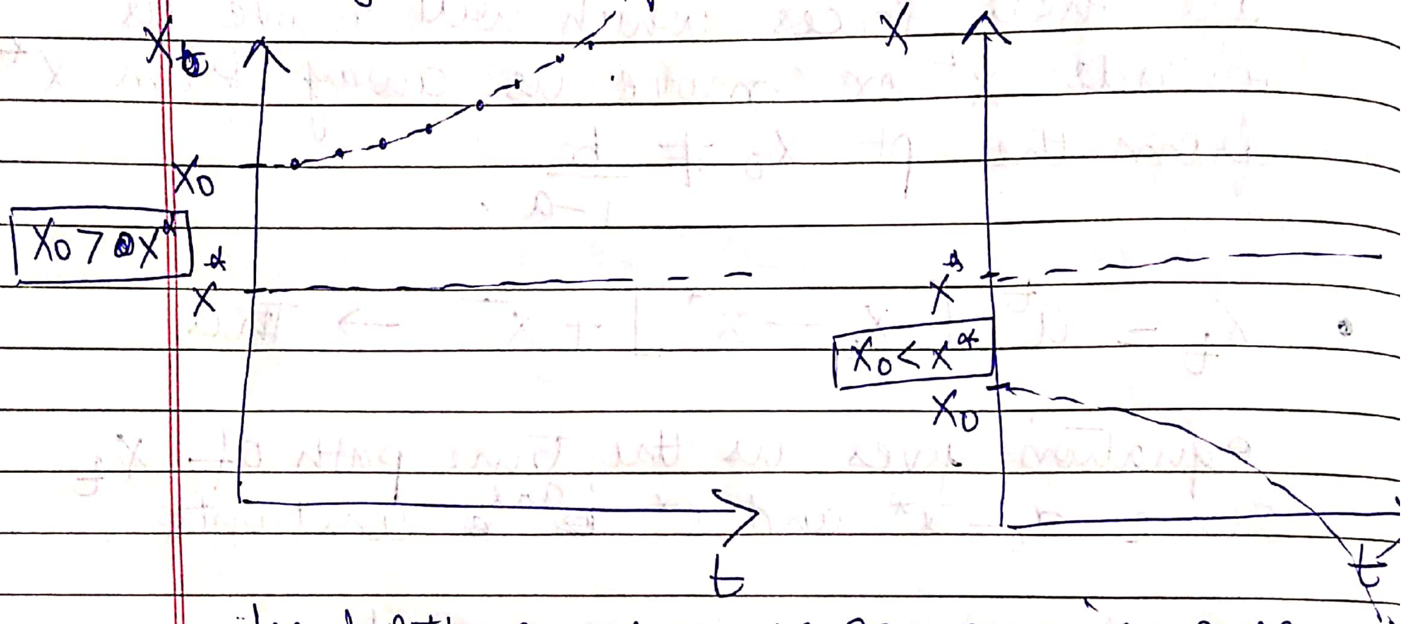
If $a^t \to 0$ as $t \to \infty$ then $x_t \to x^*$ then we can say x^* is a stable equilibrium as overtime eventually we will reach x^*.

→ • whether $a^t \to 0$ as $t \to \infty$ depends on the value of a
Since $t \in N$, $a \in \mathbb{R}$.

case(i) If $a>1$, $a^t \to \infty$ as $t \to \infty$ [Recall exponential distribution with base greater than 1]

So $x_t \to \infty / -\infty$ if $x_0 \gtrless x^*$

Unstable equilibrium, Non-oscillatory divergent time path.



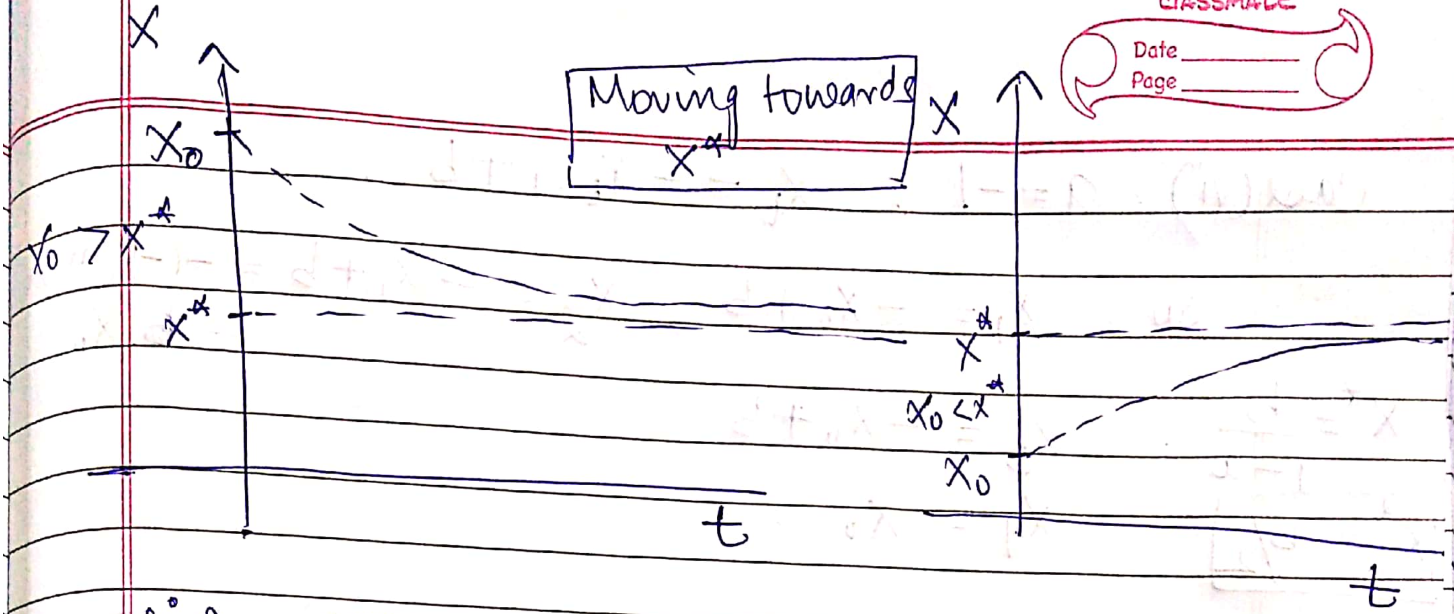
In both graphs we are moving away from x^*.

Note: The graph can also be drawn below the t axis if x_0 and x^* takes negative values. [for all cases we are going to discuss]

case(ii) $a=1 \to$ already talked about before.

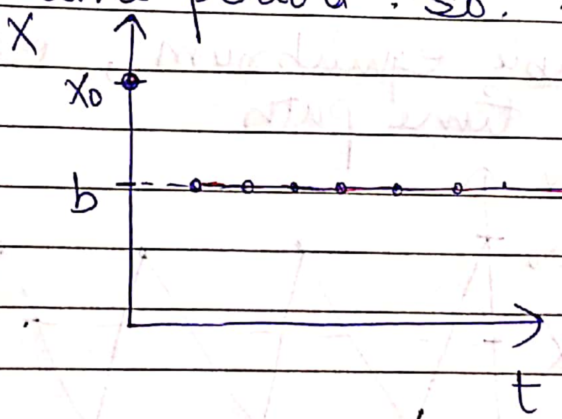
case(iii) $0<a<1$, $a^t \to 0$ as $t \to \infty$.

So. $x_t \to x^*$ stable equilibrium, Non oscillatory convergent time path.



case (iv) If $a = 0$, then $X_t = b \;\forall t$

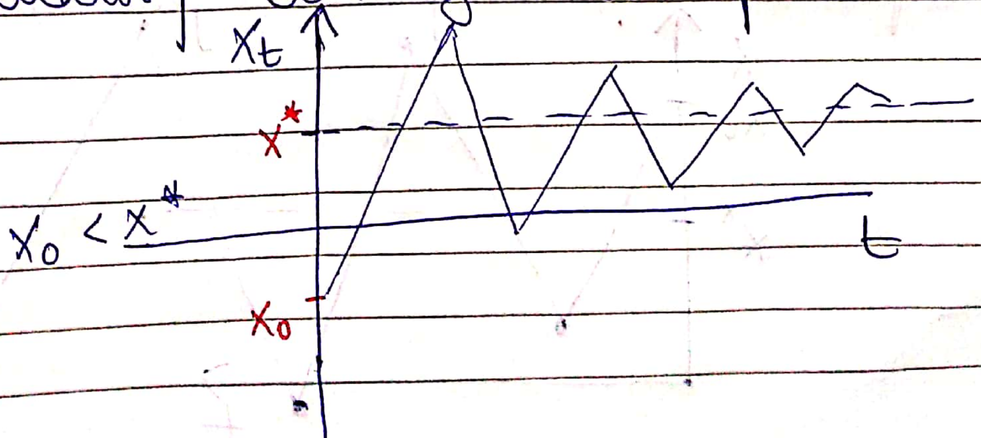
$X^* = b$, variable X is a constant b for all time period. So, stable eq^m



case (v) $-1 < a < 0$ $a^t \to 0$ as $t \to \infty$.

[Recall our discussion of geometric series from previous semester].

So, $X_t \to X^*$, stable eq^m, damped oscillatory convergent time path.



Case (vi) $a = -1$, $X_t = -X_{t-1} + b$.

So. $X_1 = -X_0 + b$, $X_2 = -X_1 + b = -(-X_0 + b) + b = X_0$

$X^* = \frac{b}{1-a}$

$\boxed{X^* = b/2}$

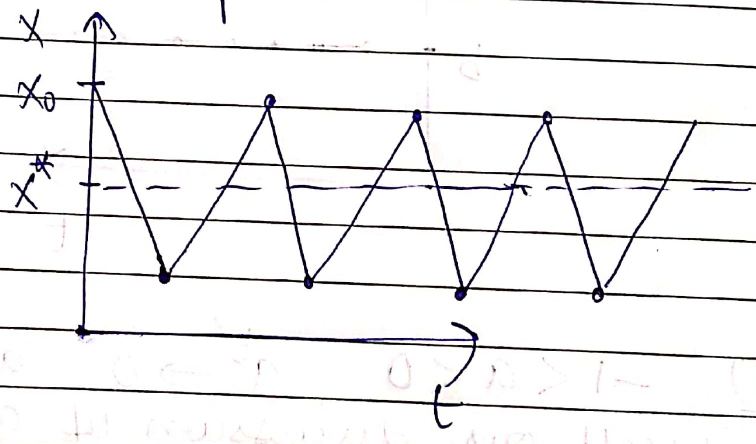
$X_3 = -X_0 + b$

$X_4 = X_0$

⋮

So. the values of X keep on fluctuating between X_0 and $-X_0 + b$! It will never reach the equilibrium value of $X^* = b/2$.

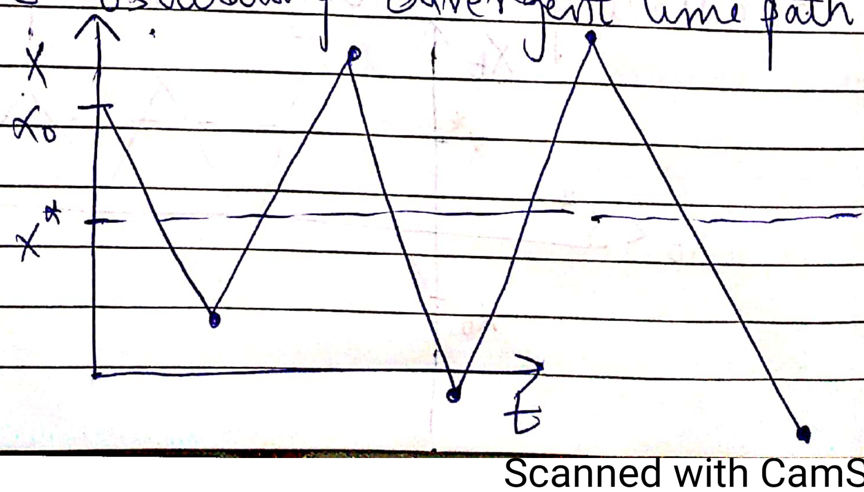
So. unstable equilibrium, equal oscillatory divergent time path



Case (vii) $a < -1$, $a^t \to \infty / -\infty$ as $t \to \infty$

So. $X^t \to \infty / -\infty$, unstable eqm

explosive oscillatory divergent time path.



Conclusion

(i) $|a| < 1$ stable equilibrium

$|a| \geq 1$ unstable equilibrium.

(ii) $a > 0 \rightarrow$ non oscillatory time path

$a < 0 \rightarrow$ oscillatory time path

Example

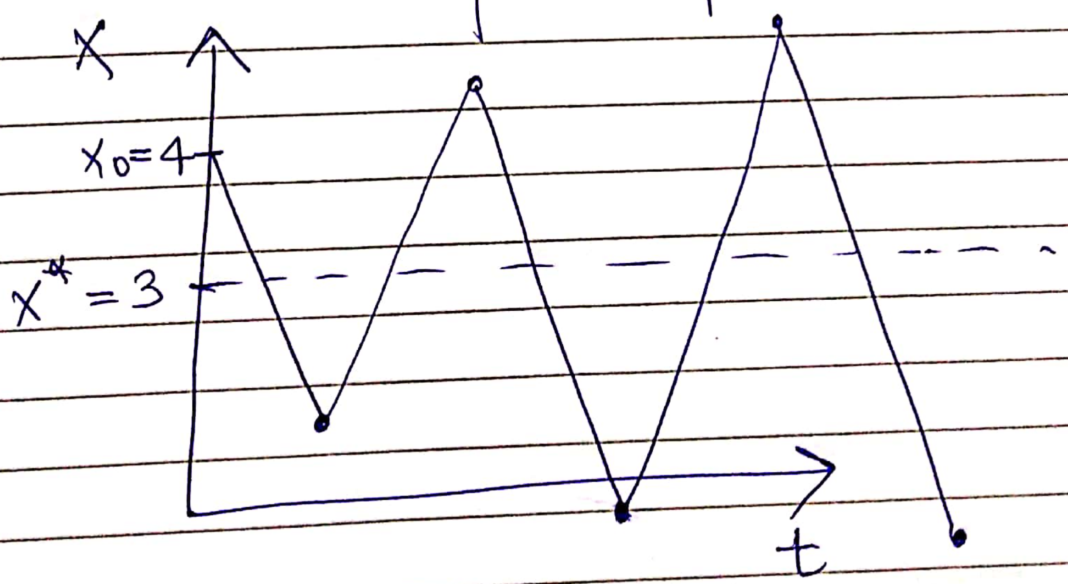
(i) $x_t + 2x_{t-1} = 9 \quad [x_0 = 4]$

or $x_t = -2x_{t-1} + 9 \quad \left[\begin{matrix} a = -2 \\ b = 9 \end{matrix} \right]$

Solution $x_t = (-2)^t \left[4 - \frac{9}{1-(-2)} \right] + \frac{9}{1-(-2)}$

$\boxed{x_t = (-2)^t + 3}$ Equilibrium $x^* = \frac{b}{1-a}$ $\boxed{x^* = 3}$

as $a = -2 < -1$ unstable equilibrium, explosive oscillatory time path.



classmate

Date
Page

(2) The population of an island is currently 50,000. It is declining at 1% per annum. However, there is net immigration of 5,000 persons each year. Write the difference equation for the population after t years and solve it. Find the steady state equilibrium and depict how the solution converges to / diverges away from the equilibrium state.

Ans $P_t = 5000 + (1-0.01) P_{t-1}$

$P_t = 0.99 P_{t-1} + 5000$ is the difference equation

$a = 0.99$; $b = 5000$

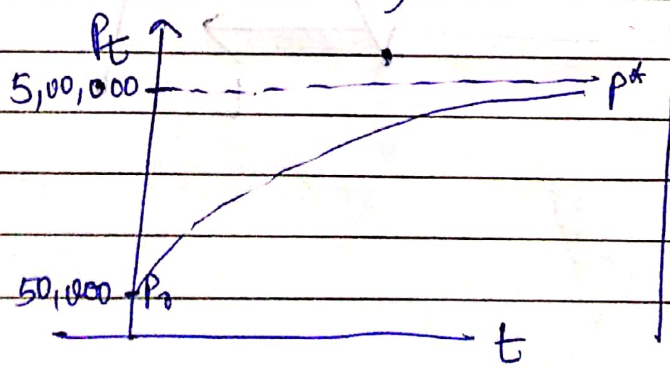
$\frac{b}{1-a} = \frac{5000}{1-0.99} = 5,00,000$; $P_0 = 50,000$

solution $P_t = (0.99)^t \left[50,000 - \frac{5000}{1-0.99} \right] + \frac{5000}{1-0.99}$

$P_t = (0.99)^t [-4,50,000] + 5,00,000$

Steady state equilibrium $P^* = \frac{b}{1-a} = 5,00,000$

Since $a = 0.99 \Rightarrow 0 < a < 1 \Rightarrow$ monotonic convergent path of P_t



Scanned with CamScanner

Date
Page

Economic Application of Difference equations

The Hog Cycle : A cobweb Model

Let there are N identical firms ; Each firm behaves competitively taking price P as given. Each firm wants to maximize their profits.

Cost f^n : $C(q_i) = \alpha q_i + \beta q_i^2$

Demand f^n : $D_i(p) = q_i = \gamma - \delta P_i$

$\alpha, \beta, \gamma, \delta > 0 \qquad i = 1, 2, --- N$

Profit function of firm i.

$$\Pi_i = \bar{P} q_i - c(q_i) = P q_i - \alpha q_i - \beta q_i^2$$

Max profit $\frac{d\Pi}{dq_i} = \bar{P} - \alpha - 2\beta q_i = 0$

$$\frac{d^2\Pi}{dq_i^2} = -2\beta < 0.$$

$$q_i = \frac{\bar{P} - \alpha}{2\beta} = \frac{P - \alpha}{2\beta} \rightarrow \text{Supply function of firm } i \text{ (Provided } P > \alpha)$$

Since all firms are identical. Every firm i will have the same maximization exercise and will have same supply function

So Total supply from all N firms is

$$S = \frac{N(P-\alpha)}{2\beta} \qquad (P > \alpha)$$

Now suppose it takes 1 period to supply the product. This means short run supply is fixed in any period. Firms when deciding how to supply in period ~~t~~ ~~[illegible]~~ remembers the price P_{t-1} at time $t-1$ and expects P_t to be the same as P_{t-1}. So supply S_t in period t will depend on P_{t-1} [as supply increases with a lag]

$$S_t = \frac{N(P_{t-1}-\alpha)}{2\beta}.$$

$$D_t = \nu - \delta P_t$$ [Demand in period t depends on price in period t]

Eq^m $\quad S_t = D_t$

$$\frac{N(P_{t-1}-\alpha)}{2\beta} = \nu - \delta P_t$$

$$\Rightarrow \boxed{P_t = \frac{-N}{2\beta\delta} P_{t-1} + \frac{\alpha N + 2\beta\nu}{2\beta\delta}}$$

So, we have a difference eq^n now.

$a = \frac{-N}{2\beta\delta}$; $b = \frac{\alpha N + 2\beta\nu}{2\beta\delta}$

Equilibrium state $P^* = \frac{b}{1-a} = \frac{\alpha N + 2\beta\nu}{2\beta\delta + N}$.

$$Q^* = \nu - \delta P^* = \frac{N\nu - \delta\alpha N}{2\beta\delta + N}$$

For stable equilibrium

As $a = \frac{-N}{2\beta\delta} < 0$.

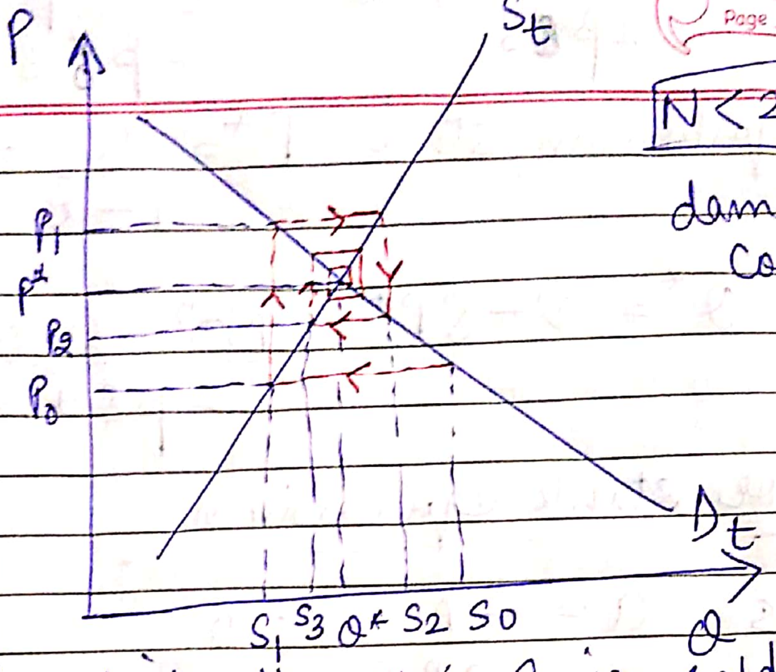
(A) So, for stability $-1 < a < 0$.

or $\frac{-N}{2\beta\delta} > -1 \iff \frac{N}{2\beta\delta} < 1 \iff$

$$\frac{N}{2\beta} < \delta.$$

$\downarrow$

| Slope of Supply curve | < | Slope of Demand curve |



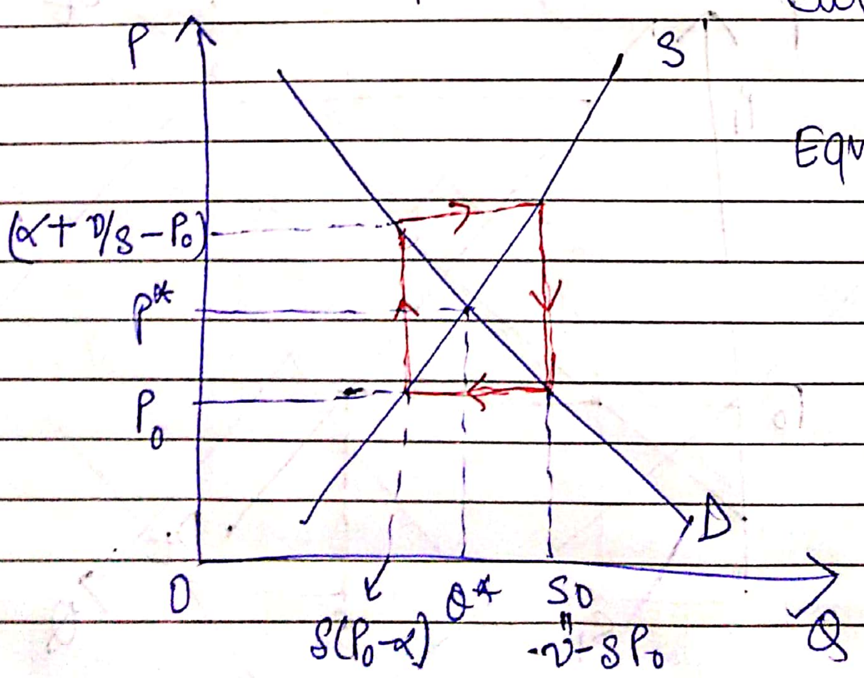
$\boxed{N<2\beta\delta}$

damped cobweb.

Let S_0 is the Supply at time period 0. The price at which all of this can be sold is P_0. Supply S_1 in period 1 will depend on P_0. The price at which all of this S_1 is sold is P_1 (from demand f^n). If price is P_1, supply in period 2 is S_2 (depends on P_1) [from supply f^n] and so on. The resulting price cycles are damped and both price and quantity converges to a steady state eqm (P^*, Q^*).

(B) $N = 2\beta\delta \Rightarrow a = -1$: unstable equilibrium

|Slope of supply curve| = |slope of Demand curve|



Equal oscillatory cobweb

The pair (S_t, P_t) oscillates perpetually between two values $(\gamma - \delta P_0, P_0)$ and $(\delta(P_0 - \alpha), \alpha + \gamma/\delta - P_0)$

Ⓒ $N > 2\beta\delta \Rightarrow a < -1$ unstable equilibrium

$|\text{slope of supply curve}| > |\text{slope of Demand curve}|$